DAY SEVEN

Binomial Theorem and Mathematical Induction

Learning & Revision for the Day

- Binomial Theorem
- Binomial Theorem for Positive Index
- Properties of Binomial Coefficient
- Applications of Binomial Theorem
- Binomial Theorem for Negative/Rational Index
- Principle of Mathematical Induction

Binomial Theorem

Binomial theorem describes the algebraic expansion of powers of a binomial. According to this theorem, it is possible to expand $(x + y)^n$ into a sum involving terms of the form $ax^b y^c$, where the exponents b and c are non-negative integers with $b + c = n$. The coefficient a of each term is a specific positive integer depending on n and b, is known as the binomial coefficient $\binom{n}{b}$.

Binomial Theorem for Positive Index

An algebraic expression consisting of two terms with (+) ve or (–)ve sign between them, is called binomial expression.

If n is any positive integer,

then $(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + \cdots + {}^nC_n a^n$

$= \sum_{r=0}^{n} {}^nC_r \cdot x^{n-r} a^r$, where x and a are **real** (complex) **numbers**.

(i) The coefficient of terms equidistant from the beginning and the end, are equal.

(ii) $(x - a)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} a + \cdots + (-1)^n \, {}^nC_n a^n$

(iii) $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$

(iv) Total number of terms in the expansion $(x + a)^n$ is $(n + 1)$.

(v) If n is a positive integer, then the number of terms in $(x + y + z)^n$ is $\frac{(n+1)(n+2)}{2}$.

(vi) The number of terms in the expansion of

$$(x+a)^n+(x-a)^n=\begin{cases}\frac{n+2}{2}, & \text{if } n \text{ is even}\\ \frac{n+1}{2}, & \text{if } n \text{ is odd}\end{cases}$$

(vii) The number of terms in the expansion of

$$(x+a)^n-(x-a)^n=\begin{cases}\frac{n}{2}, & \text{if } n \text{ is even}\\ \frac{n+1}{2}, & \text{if } n \text{ is odd}\end{cases}$$

General Term and Middle Term

(i) Let $(r+1)$th term be the **general term** in the expansion of $(x+a)^n$.

$$T_{r+1}={}^nC_r x^{n-r}a^r$$

(ii) If expansion is $(x-a)^n$, then the **general term** is

$$(-1)^r \cdot {}^nC_r\, x^{n-r}a^r.$$

(iii) The **middle term** in the expansion of $(a+x)^n$.

(a) ***Case I*** If n is even, then $\left(\frac{n}{2}+1\right)$th term is middle term.

(b) ***Case II*** If n is odd, then $\frac{(n+1)}{2}$th term and $\frac{(n+3)}{2}$th terms are middle terms.

(iv) $(p+1)^{\text{th}}$ term from end $=(n-p+1)^{\text{th}}$ term from beginning.

(v) For making a term independent of x we put $r=n$ in general term of $(x+a)^n$, so we get ${}^nC_na^n$, that is independent of x.

NOTE If the coefficients of rth, $(r+1)$th, $(r+2)$th term of $(1+x)^n$ are in AP, then $n^2-(4r+1)n+4r^2=2$

Greatest Term

If T_r and T_{r+1} be the rth and $(r+1)$th terms in the expansion of $(1+x)^n$, then

$$\frac{T_{r+1}}{T_r}=\frac{{}^nC_r\cdot x^r}{{}^nC_{r-1}\cdot x^{r-1}}=\frac{n-r+1}{r}\cdot x$$

Let numerically, T_{r+1} be the greatest term in the above expansion. Then, $T_{r+1}\geq T_r$ or $\frac{T_{r+1}}{T_r}\geq 1$.

$$\therefore \quad \frac{n-r+1}{r}|x|\geq 1 \text{ or } r\leq \frac{(n+1)}{(1+|x|)}|x| \qquad \ldots(\text{i})$$

(i) Now, substituting values of n and x in Eq. (i), we get $r\leq m+f$ or $r\leq m$, where m is a positive integer and f is a fraction such that $0<f<1$.

(ii) When $r\leq m+f$, T_{m+1} is the greatest term, when $r\leq m$, T_m and T_{m+1} are the greatest terms and both are equal.

(iii) The coefficients of the middle terms in the expansion of $(a+x)^n$ are called **greatest coefficients.**

Properties of Binomial Coefficients

In the binomial expansion of $(1+x)^n$,

$(1+x)^n={}^nC_0+{}^nC_1\cdot x+{}^nC_2\cdot x^2+\ldots+{}^nC_r\cdot x^r+\ldots{}^nC_n\cdot x^n$,

where, ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ are the coefficients of various powers of x are called **binomial coefficients** and it is also written as $C_0, C_1, \ldots, C_n$ or $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

- ${}^nC_r={}^nC_{n-r}$
- ${}^nC_{r_1}={}^nC_{r_2}\Rightarrow r_1=r_2$ or $r_1+r_2=n$
- ${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r$
- $\frac{{}^nC_r}{{}^nC_{r-1}}=\frac{n-r+1}{r}$
- $r\cdot{}^nC_r=n\cdot{}^{n-1}C_{r-1}$
- $\frac{{}^nC_r}{r+1}=\frac{{}^{n+1}C_{r+1}}{n+1}$
- $C_0+C_1+C_2+\ldots+C_n=2^n$
- $C_0+C_2+C_4+\ldots=C_1+C_3+C_5+\ldots=2^{n-1}$
- $C_0-C_1+C_2-C_3+\ldots+(-1)^n\cdot C_n=0$
- $C_0^2+C_1^2+C_2^2+\ldots+C_n^2={}^{2n}C_n=\frac{(2n)!}{(n!)^2}$
- $C_0^2-C_1^2+C_2^2-C_3^2+\ldots=\begin{cases}(-1)^{n/2}\cdot{}^nC_{n/2}, & \text{if } n \text{ is even}\\ 0, & \text{if } n \text{ is odd}\end{cases}$
- $C_0\cdot C_r+C_1\cdot C_{r+1}+\ldots+C_{n-r}\cdot C_n={}^{2n}C_{n-r}=\frac{(2n)!}{(n-r)!(n+r)!}$
- $C_1-2C_2+3C_3-\ldots=0$
- $C_0+2C_1+3C_2+\ldots+(n+1)\cdot C_n=(n+2)2^{n-1}$
- $C_0-C_2+C_4-C_6+\ldots=\sqrt{2^n}\cdot\cos\frac{n\pi}{4}$
- $C_1-C_3+C_5-C_7+\ldots=\sqrt{2^n}\cdot\sin\frac{n\pi}{4}$

Applications of Binomial Theorem

1. R-f Factor Relation

Here, we are going to discuss problems involving $(\sqrt{A}+B)^n=I+f$, where I and n are positive integers $0\leq f\leq 1, |A-B^2|=k$ and $|\sqrt{A}-B|<1$.

2. Divisibility Problem

In the expansion, $(1+\alpha)^n$. We can conclude that, $(1+\alpha)^n-1$ is divisible by α, i.e. it is a multiple of α.

3. Differentiability Problem

Sometimes to generalise the result we use the differentiation.

$$(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_nx^n$$

On differentiating w.r.t. x, we get

$n(1+x)^{n-1} = 0 + {}^nC_1 + 2 \cdot x \cdot {}^nC_2 + \ldots + n \cdot {}^nC_n \cdot x^{n-1}$

Put $x = 1$, we get, $n2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + \ldots + n\,{}^nC_n$

Binomial Theorem for Negative/Rational Index

Let n be a rational number and x be a real number such that $|x| < 1$, then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

- If n is a positive integer, then $(1+x)^n$ contains $(n+1)$ terms i.e. a finite number of terms. When n is any negative integer or rational number, then expansion of $(1+x)^n$ contains infinitely many terms.
- When n is a positive integer, then expansion of $(1+x)^n$ is valid for all values of x. If n is any negative integer or rational number, then expansion of $(1+x)^n$ is valid for the values of x satisfying the condition $|x| < 1$.
  (i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$
  (ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
  (iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$
  (iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

Principle of Mathematical Induction

In algebra, there are certain results that are formulated in terms of n, where n is a positive integer. Such results can be proved by a specific technique, which is known as the principle of mathematical induction.

First Principle of Mathematical Induction

It consists of the following three steps

***Step* I** Actual verification of the proposition for the starting value of i.

***Step* II** Assuming the proposition to be true for k, $k \geq i$ and proving that it is true for the value $(k+1)$ which is next higher integer.

***Step* III** To combine the above two steps. Let $p(n)$ be a statement involving the natural number n such that

(i) $p(1)$ is true i.e. $p(n)$ is true for $n = 1$.

(ii) $p(m+1)$ is true, whenever $p(m)$ is true

i.e. $p(m)$ is true $\Rightarrow p(m+1)$ is true. Then, $p(n)$ is true for all natural numbers n.

Product of r consecutive integers is divisible by $r!$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If $(1+ax)^n = 1 + 8x + 24x^3 + \ldots$, then the values of a and n are

(a) 2, 4 (b) 2, 3 (c) 3, 6 (d) 1, 2

2 The coefficient of x^n in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ are in the ratio → **NCERT Exemplar**

(a) 1 : 2 (b) 1 : 3
(c) 3 : 1 (d) 2 : 1

3 The value of $(1.002)^{12}$ upto fourth place of decimal is

(a) 1.0242 (b) 1.0245
(c) 1.0004 (d) 1.0254

4 The coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^n$ is

(a) ${}^nC_4$ (b) ${}^nC_4 + {}^nC_2$
(c) ${}^nC_4 + {}^nC_2 + {}^nC_2$ (d) ${}^nC_4 + {}^nC_2 + {}^nC_1 \cdot {}^nC_2$

5 If the middle term of $\left(\frac{1}{x} + x\sin x\right)^{10}$ is equal to $7\frac{7}{8}$, then the value of x is → **NCERT Exemplar**

(a) $2n\pi + \frac{\pi}{6}$ (b) $n\pi + \frac{\pi}{6}$

(c) $n\pi + (-1)^n \frac{\pi}{6}$ (d) $n\pi + (-1)\frac{\pi}{3}$

6 If the 7th term in the binomial expansion of $\left(\frac{3}{\sqrt[3]{84}} + \sqrt{3}\ln x\right)^9$, $x > 0$ is equal to 729, then x can be → **JEE Mains 2013**

(a) e^2 (b) e (c) $e/2$ (d) $2e$

7 If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$, $x \neq 0$ is 28, then the sum of the coefficients of all the terms in this expansion, is → **JEE Mains 2016**

(a) 64 (b) 2187 (c) 243 (d) 729

8 In the binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ is equal to

(a) $\frac{5}{n-4}$ (b) $\frac{6}{n-5}$
(c) $\frac{n-5}{6}$ (d) $\frac{n-4}{5}$

9 In the expansion of the following expression $1 + (1+x) + (1+x)^2 + \ldots + (1+x)^n$, the coefficient of $x^4 (0 \leq k \leq n)$ is

(a) ${}^{n+1}C_{k+1}$ (b) ${}^nC_k$
(c) ${}^nC_{n-k-1}$ (d) None of these

10 The coefficient of t^{24} in the expansion of $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is

(a) ${}^{12}C_6+2$ (b) ${}^{12}C_5$ (c) ${}^{12}C_6$ (d) ${}^{12}C_7$

11 The coefficient of x^{53} in the following expansion $\sum_{m=0}^{100} {}^{100}C_m(x-3)^{100-m}\cdot 2^m$ is

(a) ${}^{100}C_{47}$ (b) ${}^{100}C_{53}$ (c) $-{}^{100}C_{53}$ (d) $-{}^{100}C_{100}$

12 If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, then the value of p is

→ **NCERT Exemplar**

(a) ± 3 (b) ± 1
(c) ± 2 (d) None of these

13 The constant term in the expansion of $\left(1+x+\frac{2}{x}\right)^6$, is

(a) 479 (b) 517 (c) 569 (d) 581

14 If in the expansion of $(1+x)^m(1-x)^n$, the coefficient of x and x^2 are 3 and –6 respectively, them m is

(a) 6 (b) 9 (c) 12 (d) 24

15 If n is a positive integer, then $(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}$ is

(a) an irrational number → **AIEEE 2012**
(b) an odd positive integer
(c) an even positive integer
(d) a rational number other than positive integers

16 If the $(r+1)$ th term in the expansion of $\left(\sqrt[3]{\frac{a}{\sqrt{b}}}+\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{21}$ has the same power of a and b, then the value of r is

(a) 9 (b) 10 (c) 8 (d) 6

17 If x^{2k} occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$, then

(a) $n-2k$ is a multiple of 2 (b) $n-2k$ is a multiple of 3
(c) $k=0$ (d) None of these

18 The ratio of the coefficient of x^{15} to the term independent of x in the expansion of $\left(x^2+\frac{2}{x}\right)^{15}$, is

→ **JEE Mains 2013**

(a) 7 : 16 (b) 7 : 64 (c) 1 : 4 (d) 1 : 32

19 The greatest term in the expansion of $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$ is

(a) $\binom{20}{7}\frac{1}{27}$ (b) $\binom{20}{6}\frac{1}{81}$
(c) $\frac{1}{9}\binom{20}{9}$ (d) None of these

20 The largest term in the expansion of $(3+2x)^{50}$, where $x=\frac{1}{5}$ is

(a) 5th (b) 3th (c) 7th (d) 6th

21 If the sum of the coefficients in the expansion of $(x-2y+3z)^n$ is 128, then the greatest coefficient in the expansion of $(1+x)^n$ is

(a) 35 (b) 20 (c) 10 (d) None of these

22. If for positive integers $r>1$, $n>2$, the coefficient of the $(3r)$th and $(r+2)$th powers of x in the expansion of $(1+x)^{2n}$ are equal, then

(a) $n=2r$ (b) $n=3r$
(c) $n=2r+1$ (d) None of these

23 If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ is equal to

(a) $(n-1)a_n$ (b) na_n
(c) $\frac{1}{2}na_n$ (d) None of these

24 $\sum_{r=0}^{n}(-1)^r({}^nC_r)\frac{1+rx}{1+nx}$ is equal to

(a) 1 (b) –1 (c) n (d) 0

25 $\binom{30}{0}\binom{30}{10}-\binom{30}{1}\binom{30}{11}+\ldots+\binom{30}{20}\binom{30}{30}$ is equal to

(a) ${}^{30}C_{11}$ (b) ${}^{60}C_{10}$ (c) ${}^{30}C_{10}$ (d) ${}^{65}C_{55}$

26 The value of $({}^{21}C_1-{}^{10}C_1)+({}^{21}C_2-{}^{10}C_2)+({}^{21}C_3-{}^{10}C_3)+({}^{21}C_4-{}^{10}C_4)+\ldots+({}^{21}C_{10}-{}^{10}C_{10})$ is

→ **JEE Mains 2017**

(a) $2^{21}-2^{11}$ (b) $2^{21}-2^{10}$ (c) $2^{20}-2^9$ (d) $2^{20}-2^{10}$

27 The sum of the series ${}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+\ldots+{}^{20}C_{10}$ is → **AIEEE 2007**

(a) $-{}^{20}C_{10}$ (b) $\frac{1}{2}{}^{20}C_{10}$ (c) 0 (d) ${}^{20}C_{10}$

28 If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nX^n$, then the value of $C_0+2C_1+3C_2+\ldots+(n+1)C_n$ will be

(a) $(n+2)2^{n-1}$ (b) $(n+1)2^n$
(c) $(n+1)2^{n-1}$ (d) $(n+2)2^n$

29 If $n>(8+3\sqrt{7})^{10}, n\in N$, then the least value of n is

(a) $(8+3\sqrt{7})^{10}-(8-3\sqrt{7})^{10}$
(b) $(8+3\sqrt{7})^{10}+(8-3\sqrt{7})^{10}$
(c) $(8+3\sqrt{7})^{10}-(8-3\sqrt{7})^{10}+1$
(d) $(8+3\sqrt{7})^{10}-(8-3\sqrt{7})^{10}-1$

30 $49^n+16n-1$ is divisible by

(a) 3 (b) 19 (c) 64 (d) 29

31 If $A=1000^{1000}$ and $B=(1001)^{999}$, then

(a) $A>B$ (b) $A=B$
(c) $A<B$ (d) None of these

32 If ${}^{n-1}C_r=(k^2-3)\cdot{}^nC_{r+1}$, then k belongs to

(a) $(-\infty,-2]$ (b) $[2,\infty)$ (c) $[-\sqrt{3},\sqrt{3}]$ (d) $(\sqrt{3},2]$

33 The remainder left out when $8^{2n}-(62)^{2n+1}$ is divided by 9, is

(a) 0 (b) 2 (c) 7 (d) 8

34 If x is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is → **AIEEE 2003**

(a) 7th term (b) 5th term (c) 8th term (d) 6th term

35 Let $P(n): n^2+n+1$ ($n \in N$) is an even integer. Therefore, $P(n)$ is true

(a) for $n>1$ (b) for all n (c) for $n>2$ (d) None of these

36 For all $n \in N$, $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n!$ is equal to → **NCERT Exemplar**

(a) $(n+1)!-2$ (b) $(n+1)!$
(c) $(n+1)!-1$ (d) $(n+1)!-3$

37 For each $n \in N$, $2^{3n}-1$ is divisible by

(a) 8 (b) 16
(c) 32 (d) None of these

38 Let $S(k) = 1+3+5+\ldots+(2k-1) = 3+k^2$. Then, which of the following is true? → **AIEEE 2004**

(a) $S(1)$ is correct
(b) $S(k) \Rightarrow S(k+1)$
(c) $S(k) \nRightarrow S(k+1)$
(d) Principle of mathematical induction can be used to prove the formula

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 The coefficient of x^{2m+1} in the expansion of $E = \dfrac{1}{(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2m})}, |x|<1$ is

(a) 3 (b) 2 (c) 1 (d) 0

2 $C_1 - \dfrac{C_2}{2} + \dfrac{C_3}{3} - \ldots + (-1)^{n-1}\dfrac{C_n}{n}$ is equal to

(a) $1 - \dfrac{1}{2} + \dfrac{1}{3} - \ldots + \dfrac{(-1)^{n-1}}{n}$ (b) $1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}$

(c) $1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n-1}$ (d) None of these

3 If the coefficient of x^5 in $\left[ax^2 + \dfrac{1}{bx}\right]^{10}$ is a times and equal to the coefficient of x^{-5} in $\left[ax - \dfrac{1}{b^2x^2}\right]^{10}$, then the value of ab is

(a) $(b)^{-3}$ (b) $-(b)^6$ (c) $(b)^{-1}$ (d) None of these

4 The sum of coefficients of integral powers of x in the binomial expansion of $(1-2\sqrt{x})^{50}$, is → **JEE Mains 2015**

(a) $\dfrac{1}{2}(3^{50}+1)$ (b) $\dfrac{1}{2}(3^{50})$

(c) $\dfrac{1}{2}(3^{50}-1)$ (d) $\dfrac{1}{2}(2^{50}+1)$

5 The term independent of x in expansion of $\left(\dfrac{x+1}{x^{2/3}-x^{1/3}+1} - \dfrac{x-1}{x-x^{1/2}}\right)$ is → **JEE Mains 2013**

(a) 4 (b) 120 (c) 210 (d) 310

6 If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then $C_0^2 + C_1^2 + C_2^2 + C_3^2 + \ldots + C_n^2$ is equal to

(a) $\dfrac{n!}{n!n!}$ (b) $\dfrac{(2n)!}{n!n!}$

(c) $\dfrac{(2n)!}{n!}$ (d) None of these

7 If a and d are two complex numbers, then the sum to $(n+1)$ terms of the following series $aC_0 - (a+d)C_1 + (a+2d)C_2 - \ldots + \ldots$ is

(a) $\dfrac{a}{2^n}$ (b) na

(c) 0 (d) None of these

8 $\displaystyle\sum_{p=1}^{n}\sum_{m=p}^{n}\binom{n}{m}\binom{m}{p}$ is equal to

(a) 3^n (b) 2^n
(c) 3^n+2^n (d) 3^n-2^n

9 The sum of the series

$\displaystyle\sum_{r=0}^{n}(-1)^r\, {}^nC_r\left(\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \ldots + m \text{ terms}\right)$ is

(a) $\dfrac{2^{mn}-1}{2^{mn}(2^n-1)}$ (b) $\dfrac{2^{mn}-1}{2^n-1}$

(c) $\dfrac{2^{mn}+1}{2^n+1}$ (d) None of these

10 The value of x, for which the 6th term in the expansion of $\left\{2^{\log_2\sqrt{9^{x-1}+7}} + \dfrac{1}{2^{(1/5)\log_2(3^{x-1}+1)}}\right\}^7$ is 84, is equal to

(a) 4 (b) 3 (c) 2 (d) 5

11 If the last term in the binomial expansion of $\left(2^{1/3} - \dfrac{1}{\sqrt{2}}\right)^n$ is $\left(\dfrac{1}{3^{5/3}}\right)^{\log_3 8}$, then the 5th term from the beginning is

(a) 210 (b) 420
(c) 105 (d) None of these

12 The sum of the coefficients of all odd degree terms in the expansion of $(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5, (x>1)$ is → **JEE Mains 2018**

(a) -1 (b) 0 (c) 1 (d) 2

13 The greatest value of the term independent of x, as α varies over R, in the expansion of $\left(x\cos\alpha + \frac{\sin\alpha}{x}\right)^{20}$ is

(a) ${}^{20}C_{10}$ (b) ${}^{20}C_{15}$ (c) ${}^{20}C_{19}$ (d) None of these

14 **Statement I** For each natural number $n, (n+1)^7 - n^7 - 1$ is divisible by 7.

Statement II For each natural number $n, n^7 - n$ is divisible by 7. **→ AIEEE 2011**

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true, Statement II is correct explanation of Statement I.
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I
(d) Statement I is true, Statement II is false

15 If the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6} : 1$, then

Statement I The value of n is 10.

Statement II $\dfrac{2^{\frac{n-4}{4}} \cdot 3^{-1}}{2 \cdot 3^{\frac{4+n}{4}}} = \sqrt{6}$ **→ NCERT Exemplar**

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

ANSWERS

SESSION 1	**1** (a)	**2** (d)	**3** (a)	**4** (d)	**5** (c)	**6** (b)	**7** (d)	**8** (d)	**9** (a)	**10** (a)
	11 (c)	**12** (c)	**13** (d)	**14** (c)	**15** (a)	**16** (a)	**17** (b)	**18** (d)	**19** (a)	**20** (c)
	21 (a)	**22** (c)	**23** (c)	**24** (d)	**25** (c)	**26** (d)	**27** (b)	**28** (a)	**29** (b)	**30** (c)
	31 (a)	**32** (d)	**33** (b)	**34** (c)	**35** (d)	**36** (c)	**37** (d)	**38** (b)		
SESSION 2	**1** (c)	**2** (b)	**3** (b)	**4** (a)	**5** (c)	**6** (b)	**7** (c)	**8** (d)	**9** (a)	**10** (c)
	11 (a)	**12** (d)	**13** (d)	**14** (b)	**15** (c)					

Hints and Explanations

1 Given that, $(1+ax)^n = 1 + 8x + 24x^2 + \ldots$

$\Rightarrow \quad 1 + \frac{n}{1}ax + \frac{n(n-1)}{1\cdot 2}a^2x^2 + \ldots$

$= 1 + 8x + 24x^2 + \ldots$

On comparing the coefficients of x, x^2, we get

$na = 8, \frac{n(n-1)}{1\cdot 2}a^2 = 24$

$\Rightarrow \quad na(n-1)a = 48$

$\Rightarrow \quad 8(8-a) = 48 \quad \Rightarrow \quad 8 - a = 6$

$\Rightarrow \quad a = 2 \quad \Rightarrow n = 4$

2 Coefficient of x^n in $(1+x)^{2n} = {}^{2n}C_n$

and coefficient of x^n in $(1+x)^{2n-1} = {}^{2n-1}C_n$

$\therefore$ Required ratio

$$= \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{\frac{(2n)!}{n!n!}}{\frac{(2n-1)!}{n!(n-1)!}} = 2:1$$

3 We have, $(1.002)^{12}$ or it can be rewritten as $(1 + 0.002)^{12}$

$\Rightarrow \quad (1.002)^{12} = 1 + {}^{12}C_1(0.002)$
$+ {}^{12}C_2(0.002)^2 + {}^{12}C_3(0.002)^3 + \ldots$

We want the answer upto 4 decimal places and as such we have left further expansion.

$\therefore (1.002)^{12} = 1 + 12(0.002)$
$+ \frac{12\cdot 11}{1\cdot 2}(0.002)^2 + \frac{12\cdot 11\cdot 10}{1\cdot 2\cdot 3}(0.002)^3 + \ldots$
$= 1 + 0.024 + 2.64\times 10^{-4} + 1.76\times 10^{-6} + \ldots$
$= 1.0242$

4 $(1 + x + x^2 + x^3)^n = \{(1+x)^n (1+x^2)^n\}$
$= (1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + {}^nC_4x^4 + \ldots {}^nC_nx^n)$
$(1 + {}^nC_1x^2 + {}^nC_2x^4 + \ldots + {}^nC_nx^{2n})$

Therefore, the coefficient of x^4

$= {}^nC_2 + {}^nC_2\,{}^nC_1 + {}^nC_4$
$= {}^nC_4 + {}^nC_2 + {}^nC_1\,{}^nC_2$

5 $\left(\frac{1}{x} + x\sin x\right)^{10}$

Here, $n = 10$ [even]

$\Rightarrow$ Middle term $= \left(\frac{10}{2} + 1\right)$th $=$ 6th

$T_6 = {}^{10}C_5\left(\frac{1}{x}\right)^{10-5}(x\sin x)^5$

$\Rightarrow \quad 252(\sin x)^5 = 7\frac{7}{8} = \frac{63}{8}$

$\Rightarrow \quad (\sin x)^5 = \frac{1}{32} \Rightarrow \sin x = \frac{1}{2}$

$\Rightarrow \quad \sin x = \sin \pi/6$

$\therefore \quad x = n\pi + (-1)^n \frac{\pi}{6}$

6 $T_7 = {}^9C_6\left(\frac{3}{\sqrt[3]{84}}\right)^3 (\sqrt{3}\ln x)^6 = 729$

$\Rightarrow \quad \frac{84\times 3^3}{84}\times 3^3\times(\ln x)^6 = 729$

$= (\ln x)^6 = 1$

$\Rightarrow \quad x = e$

7 Clearly number of terms in the expansion of

$\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$ is $\frac{(n+2)(n+1)}{2}$ or ${}^{n+2}C_2$.

[assuming $\frac{1}{x}$ and $\frac{1}{x^2}$ distinct]

$\therefore \quad \frac{(n+2)(n+1)}{2}=28$

$\Rightarrow \quad (n+2)(n+1)=56=(6+1)(6+2)$

$\Rightarrow \quad n=6$

Hence, sum of coefficients

$=(1-2+4)^6=3^6=729$

8 Since, in a binomial expansion of $(a-b)^n$, $n\geq 5$, the sum of 5th and 6th terms is equal to zero.

$\therefore \quad {}^nC_4a^{n-4}(-b)^4+{}^nC_5a^{n-5}(-b)^5=0$

$\Rightarrow \quad \frac{n!}{(n-4)!4!}a^{n-4}\cdot b^4-\frac{n!}{(n-5)!5!}a^{n-5}b^5=0$

$\Rightarrow \quad \frac{n!}{(n-5)!4!}a^{n-5}\cdot b^4\left(\frac{a}{n-4}-\frac{b}{5}\right)=0$

$\Rightarrow \quad \frac{a}{b}=\frac{n-4}{5}$

9 The given expression is $1+(1+x)+(1+x)^2+\ldots+(1+x)^n$ being in GP.

Let, $S=1+(1+x)+(1+x)^2+\ldots+(1+x)^n$

$=\frac{(1+x)^{n+1}-1}{(1+x)-1}=x^{-1}[(1+x)^{n+1}-1]$

$\therefore$ The coefficient of x^k in S.

= The coefficient of x^{k+1} in $[(1+x)^{n+1}-1]$

$={}^{n+1}C_{k+1}$

10 We have, $(1+t^2)^{12}(1+t^{12})(1+t^{24})$

$=(1+{}^{12}C_1t^2+{}^{12}C_2t+\ldots+{}^{12}C_6t^{12}+\ldots+{}^{12}C_{12}t^{24}+\ldots)(1+t^{12}+t^{24}+t^{36})$

$\therefore$ Coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$

$={}^{12}C_6+{}^{12}C_{12}+1={}^{12}C_6+2$

11 The given sigma expansion

$\sum_{m=0}^{100}{}^{100}C_m(x-3)^{100-m}\cdot 2^m$ can be written

as $[(x-3)+2]^{100}=(x-1)^{100}=(1-x)^{100}$

$\therefore$ Coefficient of x^{53} in

$(1-x)^{100}=(-1)^{53}\,{}^{100}C_{53}=-{}^{100}C_{53}$

12 Given expression is $\left(\frac{p}{2}+2\right)^8$

Here, $n=8$ [even]

$\Rightarrow$ Middle term $=\left(\frac{8}{2}+1\right)$th term

= 5 th term

$T_5={}^8C_4(p/2)^{8-4}(2^4)$

$\Rightarrow \quad \frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1}\times\frac{p^4}{2^4}\times 2^4=1120$

$\Rightarrow \quad p^4=16$

$\Rightarrow \quad p=\pm 2$

13 $\left(1+x+\frac{2}{x}\right)^6=1+\binom{6}{1}\left(x+\frac{2}{x}\right)+\binom{6}{2}\left(x+\frac{2}{x}\right)^2+\ldots+\binom{6}{6}\left(x+\frac{2}{x}\right)^6$

$\therefore$ Constant term

$=1+\binom{6}{2}\binom{2}{1}2^1+\binom{6}{4}\binom{4}{2}2^2+\binom{6}{6}\binom{6}{3}2^3$

$=1+60+360+160=581$

14 $(1+x)^m(1-x)^n$

$=\left\{1+mx+\frac{m(m-1)x^2}{2!}+\ldots\right\}\left[1-nx+\frac{n(n-1)}{2!}x^2-\ldots\right]$

$=1+(m-n)x+\left[\frac{n^2-n}{2}-mn+\frac{(m^2-m)}{2}\right]x^2+\ldots$

Given, $m-n=3\Rightarrow n=m-3$

and $\frac{n^2-n}{2}-mn+\frac{m^2-m}{2}=-6$

$\Rightarrow \quad \frac{(m-3)(m-4)}{2}-m(m-3)+\frac{m^2-m}{2}=-6$

$\Rightarrow \quad m^2-7m+12-2m^2+6m+m^2-m+12=0$

$\Rightarrow \quad -2m+24=0\Rightarrow m=12$

15 $(\sqrt{3}+1)^{2n}={}^{2n}C_0(\sqrt{3})^{2n}+{}^{2n}C_1(\sqrt{3})^{2n-1}+{}^{2n}C_2(\sqrt{3})^{2n-2}+\ldots+{}^{2n}C_{2n}(\sqrt{3})^{2n-2n}$

$(\sqrt{3}-1)^{2n}={}^{2n}C_0(\sqrt{3})^{2n}(-1)^0+{}^{2n}C_1(\sqrt{3})^{2n-1}(-1)^1+{}^{2n}C_2(\sqrt{3})^{2n-2}(-1)^2+\ldots+{}^{2n}C_{2n}(\sqrt{3})^{2n-2n}(-1)^{2n}$

Adding both the binomial expansions above, we get

$(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}=2[{}^{2n}C_1(\sqrt{3})^{2n-1}+{}^{2n}C_3(\sqrt{3})^{2n-3}+{}^{2n}C_5(\sqrt{3})^{2n-5}+\ldots+{}^{2n}C_{2n-1}(\sqrt{3})^{2n-(2n-1)}]$

which is most certainly an irrational number because of odd powers of $\sqrt{3}$ in each of the terms.

16 $\therefore$ General term is

$T_{r+1}={}^{21}C_r\left(\sqrt[3]{\frac{a}{\sqrt{b}}}\right)^{21-r}\left(\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^r$

$={}^{21}C_r\,a^{7-\frac{r}{2}}\cdot b^{\frac{2r}{3}-\frac{7}{2}}$

$\because$ Power of a = Power of b [given]

$\Rightarrow \quad 7-\frac{r}{2}=\frac{2}{3}r-\frac{7}{2}$

$\therefore \quad r=9$

17 The general term in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$ is given by

$T_{r+1}={}^{n-3}C_r(x)^{n-3-r}\left(\frac{1}{x^2}\right)^r$

$={}^{n-3}C_rx^{n-3-3r}$

As x^{2k} occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$, we must have

$n-3-3r=2k$ for some non-negative integer r.

$\Rightarrow \quad 3(1+r)=n-2k$

$\Rightarrow \quad n-2k$ is a multiple of 3.

18 $T_{r+1}={}^{15}C_r(x^2)^{15-r}\cdot\left(\frac{2}{x}\right)^r$

$={}^{15}C_rx^{30-2r}\cdot 2^r\cdot x^{-r}$

$={}^{15}C_r\cdot x^{30-3r}\cdot 2^r \quad \ldots(i)$

For coefficient of x^{15}, put $30-3r=15$

$\Rightarrow 3r=15\Rightarrow r=5$

$\therefore$ Coefficient of $x^{15}={}^{15}C_5\cdot 2^5$

For coefficient of independent of x

i.e. x^0 put $30-3r=0$

$\Rightarrow \quad r=10$

$\therefore$ Coefficient of $x^0={}^{15}C_{10}\cdot 2^{10}$

By condition $\Rightarrow\frac{\text{Coefficient of }x^{15}}{\text{Coefficient of }x^0}$

$=\frac{{}^{15}C_5\cdot 2^5}{{}^{15}C_{10}\cdot 2^{10}}=\frac{{}^{15}C_{10}\cdot 2^5}{{}^{15}C_{10}\cdot 2^{10}}=1:32$

19 Greatest term in the expansion of $(1+x)^n$ is T_{r+1}

where, $r=\left[\frac{(n+1)x}{1+x}\right]$

Here, $n=20$, $x=\frac{1}{\sqrt{3}}$

$\therefore \quad r=\left[\frac{21}{\sqrt{3}+1}\right]$

$=[10.5(\sqrt{3}-1)]=(7.69)\approx 7$

Hence, greatest term is

$\sqrt{3}\binom{20}{7}\left(\frac{1}{\sqrt{3}}\right)^7=\binom{20}{7}\frac{1}{27}.$

20 $\because \quad (3+2x)^{50}=3^{50}\left(1+\frac{2x}{3}\right)^{50}$

Here, $T_{r+1}=3^{50}\,{}^{50}C_r\left(\frac{2x}{3}\right)^r$

and $T_r=3^{50}\,{}^{50}C_{r-1}\left(\frac{2x}{3}\right)^{r-1}$

But $x=\frac{1}{5}$ (given)

$\therefore \quad \frac{T_{r+1}}{T_r}\geq 1\Rightarrow\frac{{}^{50}C_r}{{}^{50}C_{r-1}}\cdot\frac{2}{3}\cdot\frac{1}{5}\geq 1$

$\Rightarrow \quad 102-2r\geq 15r\Rightarrow r\leq 6$

21 Sum of the coefficients in the expansion of

$(x-2y+3z)^n$ is $(1-2+3)^n = 2^n$

(put $x = y = z = 1$)

$\therefore \quad 2^n = 128 \Rightarrow n = 7$

Therefore, the greatest coefficient in the expansion of $(1+x)^7$ is ${}^7C_3$ or ${}^7C_4$ because both are equal to 35.

22 In the expansion of $(1+x)^{2n}$, the general term $= {}^{2n}C_k x^k, 0 \le k \le 2n$

As given for $r > 1, n > 2$,

$${}^{2n}C_{3r} = {}^{2n}C_{r+2}$$

$\Rightarrow$ Either $3r = r+2$

or $3r = 2n-(r+2)$

$(\because {}^nC_x = {}^nC_y \Rightarrow x+y = n$ or $x = y)$

$\Rightarrow \quad r = 1$ or $n = 2r+1$

We take the relation only

$$n = 2r+1 \qquad (\because r > 1)$$

23 Let $b = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-(n-r)}{{}^nC_r}$

$$= n\sum_{r=0}^{n} \frac{1}{{}^nC_r} - \sum_{r=0}^{n} \frac{n-r}{{}^nC_r}$$

$$= na_n - \sum_{r=0}^{n} \frac{n-r}{{}^nC_{n-r}} \quad (\because {}^nC_r = {}^nC_{n-r})$$

$$= na_n - b \Rightarrow 2b = na_n \Rightarrow b = \frac{n}{2}a_n$$

24 Let $E = \sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{1+rx}{1+nx}\right)$

$$= \left(\frac{1}{1+nx}\right) \sum_{r=0}^{n} (-1)^r\, {}^nC_r (1+rx)$$

$$= \left(\frac{1}{1+nx}\right) \left\{ \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r + x\sum_{r=0}^{n} r(-1)^r\, {}^nC_r \right\}$$

$$= \left(\frac{1}{1+nx}\right)(0+0) = 0$$

$[\because {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots (-1)^n\, {}^nC_n = 0]$

25 Let

$$A = \binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots + \binom{30}{20}\binom{30}{30}$$

or $A = {}^{30}C_0 . {}^{30}C_{10} - {}^{30}C_1 . {}^{30}C_{11} + {}^{30}C_2 . {}^{30}C_{12} - \ldots + {}^{30}C_{20} . {}^{30}C_{30}$

= Coefficient of x^{20} in $(1+x)^{30}(1-x)^{30}$

= Coefficient of x^{20} in $(1-x^2)^{30}$

= Coefficient of x^{20} in $\sum_{r=0}^{30} (-1)^r\, {}^{30}C_r (x^2)^r$

$= (-1)^{10}\, {}^{30}C_{10}$

(for coefficient of x^{20}, let $r = 10$)

$= {}^{30}C_{10}$

26 $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$

$= ({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + \ldots + {}^{10}C_{10})$

$= \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{20}) - (2^{10}-1)$

$= \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{21} - 1) - (2^{10}-1)$

$= \frac{1}{2}(2^{21}-2) - (2^{10}-1) = 2^{20} - 1 - 2^{10} + 1$

$= 2^{20} - 2^{10}$

27 We know that,

$(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1 x + \ldots + {}^{20}C_{10}x^{10} + \ldots + {}^{20}C_{20}x^{20}$

On putting $x = -1$ in the above expansion, we get

$0 = {}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_a + {}^{20}C_{10} - {}^{20}C_{11} + \ldots + {}^{20}C_{20}$

$\Rightarrow \quad 0 = {}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_9 + {}^{20}C_{10} - {}^{20}C_9 + \ldots + {}^{20}C_0$

$\Rightarrow 0 = 2({}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_9) + {}^{20}C_{10}$

$\Rightarrow \quad {}^{20}C_{10} = 2({}^{20}C_0 - {}^{20}C_1 + \ldots + {}^{20}C_{10})$

$\Rightarrow \quad {}^{20}C_0 - {}^{20}C_1 + \ldots + {}^{20}C_{10} = \frac{1}{2}\, {}^{20}C_{10}$

28 Since, $x(1+x)^n = xC_0 + C_1x^2 + C_2x^3 + \ldots + C_nx^{n+1}$

On differentiating w.r.t. x, we get

$(1+x)^n + nx(1+x)^{n-1} = C_0 + 2C_1x + 3C_2x^2 + \ldots + (n+1)C_nx^n$

Put $x = 1$, we get

$C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = 2^n + n2^{n-1} = 2^{n-1}(n+2)$

29 Let $f = (8-3\sqrt{7})^{10}$, here $0 < f < 1$

$\therefore \quad (8+3\sqrt{7})^{10} + (8-3\sqrt{7})^{10}$ is an integer, hence this is the value of n.

30 We have,

$49^n + 16n - 1 = (1+48)^n + 16n - 1$

$= 1 + {}^nC_1(48) + {}^nC_2(48)^2 + \ldots + {}^nC_n(48)^n + 16n - 1$

$= (48n+16n) + {}^nC_2(48)^2 + {}^nC_3(48)^3 + \ldots + {}^nC_n(48)^n$

$= 64n + 8^2[{}^nC_2 \cdot 6^2 + {}^nC_3 \cdot 6^3 \cdot 8 + {}^nC_4 \cdot 6^4 8^2 + \ldots + {}^nC_n \cdot 6^n \cdot 8^{n-2}]$

Hence, $49^n + 16n - 1$ is divisible by 64.

31 Since, $\left(1+\frac{1}{n}\right)^n < 3$ for $\forall n \in N$

Now, $\frac{(1001)^{999}}{(1000)^{1000}} = \frac{1}{1001}\left(\frac{1001}{1000}\right)^{1000}$

$= \frac{1}{1001}\left(1+\frac{1}{1000}\right)^{1000} < \frac{1}{1001} \cdot 3 < 1$

$(1001)^{999} < (1000)^{1000}$

$\therefore \quad B < A$

32 Since, ${}^{n-1}C_r = (k^2-3)\frac{n}{r+1}\, {}^{n-1}C_r$

$\Rightarrow \quad k^2 - 3 = \frac{r+1}{n}$

$\Rightarrow \quad 0 < k^2 - 3 \le 1 \quad \left[\because n \ge r \Rightarrow \frac{r+1}{n} \le 1 \text{ and } n, r > 0\right]$

$\Rightarrow \quad 3 < k^2 \le 4$

Hence, $k \in [-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$

33 $8^{2n} - (62)^{2n+1} = (1+63)^n - (63-1)^{2n+1}$

$= (1+63)^n + (1-63)^{2n+1}$

$= [1 + {}^nC_1 \cdot 63 + {}^nC_2 \cdot (63)^2 + \ldots + (63)^n] + [1 - {}^{2n+1}C_1 \cdot 63 + {}^{(2n+1)}C_2 \cdot (63)^2 - \ldots + (-1)(63)^{(2n+1)}]$

$= 2 + 63[{}^nC_1 + {}^nC_2(63) + \ldots + (63)^{n-1} - {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2(63) - \ldots + (-1)(63)^{(2n)}]$

Hence, remainder is 2.

34 Since, $(r+1)$th term in the expansion of $(1+x)^{27/5}$

$$= \frac{\frac{27}{5}\left(\frac{27}{5}-1\right)\ldots\left(\frac{27}{5}-r+1\right)}{r!}x^r$$

Now, this term will be negative, if the last factor in numerator is the only one negative factor.

$\Rightarrow \quad \frac{27}{5} - r + 1 < 0 \Rightarrow \frac{32}{5} < r$

$\Rightarrow \quad 6.4 < r \Rightarrow$ least value of r is 7.

Thus, first negative term will be 8th.

35 Given, $P(n): n^2 + n + 1$

At $n = 1, P(1): 3$, which is not an even integer.

Thus, $P(1)$ is not true.

Also, $n(n+1) + 1$ is always an odd integer.

36 Let the statement $P(n)$ be defined as

$P(n): 1 \times 1! + 2 \times 2! + 3 \times 3! \ldots + n \times n! = (n+1)! - 1$

for all natural numbers n.

Note that $P(1)$ is true, since

$P(1): 1 \times 1! = 1 = 2 - 1 = 2! - 1$

Assume that $P(n)$ is true for some natural number k, i.e.

$P(k): 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + k \times k! = (k+1)! - 1 \quad \ldots(i)$

To prove $P(k+1)$ is true, we have

$P(k+1): 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + k \times k! + (k+1) \times (k+1)!$

$= (k+1)! - 1 + (k+1)! \times (k+1)$ [by Eq. (i)]

$= (k+1+1)(k+1)! - 1$

$= (k+2)(k+1)! - 1 = (k+2)! - 1$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true. Therefore, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n.

37 Now, $2^{3n} - 1 = (2^3)^n - 1 = (1+7)^n - 1$

$$= 1 + {}^nC_1 \cdot 7 + {}^nC_2 \cdot 7^2 + \dots + {}^nC_n \cdot 7^n - 1$$

$$= 7[{}^nC_1 + {}^nC_2 7 + \dots + {}^nC_n \cdot 7^{n-1}]$$

Hence, 7 divides $2^{3n} - 1$ for all $n \in N$.

38 $S(k) = 1 + 3 + 5 + \dots + (2k-1) = 3 + k^2$

Put $k = 1$ in both sides, we get

LHS = 1 and RHS = 3 + 1 = 4

$\Rightarrow$ LHS $\neq$ RHS

Put $(k+1)$ in both sides in the place of k, we get

LHS $= 1 + 3 + 5 + \dots + (2k-1) + (2k+1)$

RHS $= 3 + (k+1)^2 = 3 + k^2 + 2k + 1$

Let LHS = RHS

$$1 + 3 + 5 + \dots + (2k-1) + (2k+1) = 3 + k^2 + 2k + 1$$

$$\Rightarrow 1 + 3 + 5 + \dots + (2k-1) = 3 + k^2$$

If $S(k)$ is true, then $S(k+1)$ is also true.

Hence, $S(k) \Rightarrow S(k+1)$

SESSION 2

1 Multiplying the numerator and denominator by $1-x$, we have

$$E = \frac{1-x}{(1-x)(1+x)(1+x^2)(1+x^4)\dots(1+x^{2^m})}$$

$$= \frac{1-x}{(1-x^2)(1+x^2)(1+x^4)\dots(1+x^{2^m})}$$

$$= \frac{1-x}{(1-x^4)(1+x^4)\dots(1+x^{2^m})}$$

$$= \frac{1-x}{(1-x^{2^{m+1}})} = (1-x)(1-x^{2^{m+1}})^{-1}$$

$$= (1-x)(1+x^{2^{m+1}} + x^{2^{m+2}} + \dots)$$

$\therefore$ Coefficient of $x^{2^{m+1}}$ is 1.

2 Since, $(1-x)^n = C_0 - C_1 \cdot x + C_2 \cdot x^2 - C_3 \cdot x^3 + \dots$

$$\Rightarrow 1 - (1-x)^n = C_1 \cdot x - C_2 \cdot x^2 + C_3 \cdot x^3 - \dots$$

$$\Rightarrow \frac{1-(1-x)^n}{x} = C_1 - C_2 \cdot x + C_3 \cdot x^2 - \dots$$

$$\Rightarrow \int_0^1 (C_1 - C_2 \cdot x + C_3 \cdot x^2 - \dots)\,dx = \int_0^1 \frac{1-(1-x)^n}{1-(1-x)}\,dx$$

$$\Rightarrow \frac{C_1}{1} - \frac{C_2}{2} + \frac{C_3}{3} - \dots = \int_0^1 \frac{1-x^n}{1-x}\,dx$$

$$\left[\because \int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx\right]$$

$$= \int_0^1 (1 + x + x^2 + \dots + x^{n-1})\,dx$$

$$= \left[x + \frac{x^2}{2} + \dots + \frac{x^n}{n}\right]_0^1 = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

3 General term is

$$T_{r+1} = {}^{10}C_r \cdot (a \cdot x^2)^{10-r} \cdot \left(\frac{1}{bx}\right)^r$$

$$= {}^{10}C_r \cdot (a)^{10-r} \left(\frac{1}{b}\right)^r (x)^{20-3r}$$

Since, x^5 occurs in T_{r+1}.

$\therefore$ $20 - 3r = 5$

$\Rightarrow$ $3r = 15 \Rightarrow r = 5$

So, the coefficient of x^5 is ${}^{10}C_5 (a)^5 (b)^{-5}$.

Again, let x^{-5} occurs in T_{r+1} of

$$\left[a \cdot x - \frac{1}{b^2 \cdot x^2}\right]^{10} \text{ is } {}^{10}C_r (ax)^{10-r} \left(-\frac{1}{b^2 x^2}\right)^r$$

$$= {}^{10}C_r (a)^{10-r} \left(-\frac{1}{b^2}\right)^r (x)^{10-3r}$$

$$10 - 3r = -5 \Rightarrow 15 = 3r \Rightarrow r = 5$$

So, the coefficient of x^{-5} is $-\ {}^{10}C_5 \frac{a^5}{b^{10}}$.

According to the given condition,

$${}^{10}C_5 \frac{a^5}{b^5} = -a\ {}^{10}C_5 \frac{a^5}{b^{10}}$$

$$\Rightarrow \quad -b^5 = a \Rightarrow -b^6 = ab$$

4 Let T_{r+1} be the general term in the expansion of $(1 - 2\sqrt{x})^{50}$.

$$\therefore \quad T_{r+1} = {}^{50}C_r (1)^{50-r} (-2x^{1/2})^r$$

$$= {}^{50}C_r \cdot 2^r \cdot x^{r/2} (-1)^r$$

For the integral power of x and r should be even integer.

$$\therefore \text{Sum of coefficients} = \sum_{r=0}^{25} {}^{50}C_{2r} (2)^{2r}$$

$$= \frac{1}{2}[(1+2)^{50} + (1-2)^{50}] = \frac{1}{2}[3^{50} + 1]$$

Alternate Method

We have,

$$(1-2\sqrt{x})^{50} = C_0 - C_1 \cdot 2\sqrt{x} + C_2 (2\sqrt{x})^2 + \dots + C_{50} (2\sqrt{x})^{50} \quad \dots \text{(i)}$$

$$(1+2\sqrt{x})^{50} = C_0 + C_1 \cdot 2\sqrt{x} + C_2 (2\sqrt{x})^2 + \dots + C_{50} (2\sqrt{x})^{50} \quad \dots \text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$(1-2\sqrt{x})^{50} + (1+2\sqrt{x})^{50} = 2[C_0 + C_2 (2\sqrt{x})^2 + \dots + C_{50} (2\sqrt{x})^{50}]$$

$$\Rightarrow \frac{(1-2\sqrt{x})^{50} + (1+2\sqrt{x})^{50}}{2} = C_0 + C_2 (2\sqrt{x})^2 + \dots + C_{50} (2\sqrt{x})^{50}$$

On putting $x = 1$, we get

$$\frac{(1-2\sqrt{1})^{50} + (1+2\sqrt{1})^{50}}{2} = C_0 + C_2 (2)^2 + \dots + C_{50} (2)^{50}$$

$$\Rightarrow \frac{(-1)^{50} + (3)^{50}}{2} = C_0 + C_2 (2)^2 + \dots + C_{50} (2)^{50}$$

$$\Rightarrow \frac{1 + 3^{50}}{2} = C_0 + C_2 (2)^2 + \dots + C_{50} (2)^{50}$$

5 $$\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{(x-1)}{x - x^{1/2}}\right]^{10}$$

$$= \left[\frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}(\sqrt{x} - 1)}\right]^{10}$$

$$= \left[\frac{(x^{1/3} + 1)(x^{2/3} + 1 - x^{1/3})}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}(\sqrt{x} - 1)}\right]^{10}$$

$$= \left[(x^{1/3} + 1) - \frac{(\sqrt{x} + 1)}{\sqrt{x}}\right]^{10} = (x^{1/3} - x^{-1/2})^{10}$$

$\therefore$ The genreal term is

$$T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{-1/2})^r$$

$$= {}^{10}C_r (-1)^r x^{\frac{10-r}{3} - \frac{r}{2}}$$

For independent for x, put

$$\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow 20 - 2r - 3r = 0$$

$$\Rightarrow \quad 20 = 5r \Rightarrow r = 4$$

$$\therefore \quad T_5 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

6 We have,

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n \quad \dots \text{(i)}$$

$$\text{and} \left(1 + \frac{1}{x}\right)^n = C_0 + C_1 \frac{1}{x} + C_2 \left(\frac{1}{x}\right)^2 + \dots + C_n \left(\frac{1}{x}\right)^n \quad \dots \text{(ii)}$$

On multiplying Eqs. (i) and (ii) and taking coefficient of constant terms in right hand side $= C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2$

In right hand side $(1+x)^n \left(1 + \frac{1}{x}\right)^n$ or in $\frac{1}{x^n}(1+x)^{2n}$ or term containing x^n in $(1+x)^{2n}$. Clearly, the coefficient of x^n in $(1+x)^{2n}$ is equal to ${}^{2n}C_n = \frac{(2n)!}{n!n!}$.

7 We can write,

$aC_0 - (a+d)C_1 + (a+2d)C_2 - \dots$ upto $(n+1)$ terms

$$= a(C_0 - C_1 + C_2 - \dots) + d(-C_1 + 2C_2 - 3C_3 + \dots) \quad \dots \text{(i)}$$

We know,

$$(1-x)^n = C_0 - C_1 x + C_2 x^2 - \dots + (-1)^n C_n x^n \quad \dots \text{(ii)}$$

On differentiating Eq. (ii) w.r.t. x, we get

$$-n(1-x)^{n-1} = -C_1 + 2C_2 x - \dots + (-1)^n C_n n x^{n-1} \quad \dots \text{(iii)}$$

On putting $x = 1$ in Eqs. (ii) and (iii), we get

$$C_0 - C_1 + C_2 - \dots + (-1)^n C_n = 0 \quad \dots \text{(iv)}$$

$$\text{and } -C_1 + 2C_2 - \dots + (-1)^n C_n = 0 \quad \dots \text{(v)}$$

From Eq. (i),

$aC_0-(a+d)C_1+(a+2d)C_2-\dots+$ upto $(n+1)$ terms

$$= a\cdot 0+d\cdot 0=0$$

[from Eqs. (iv) and (v)]

8 Since, $\binom{n}{m}\binom{m}{p}=\frac{n!}{(n-m)!\,p!\,(m-p)!}$

$$=\binom{n}{p}\binom{n-p}{m-p}$$

$\therefore$ Given series can be rewritten as

$$\sum_{p=1}^{n}\sum_{m=p}^{n}\binom{n}{p}\binom{n-p}{m-p}$$

$$=\sum_{p=1}^{n}\binom{n}{p}\sum_{m=p}^{n}\binom{n-p}{m-p}$$

$$=\sum_{p=1}^{n}\binom{n}{p}\sum_{t=0}^{n-p}\binom{n-p}{t}$$

$$=\sum_{p=1}^{n}\binom{n}{p}2^{n-p} \quad [\text{put } m-p=t]$$

$$=2^n\sum_{p=1}^{n}\binom{n}{p}\cdot\frac{1}{2^p}=2^n\left[\left(1+\frac{1}{2}\right)^n-1\right]$$

$$=3^n-2^n$$

9 $\sum_{r=0}^{n}(-1)^r$

$${}^nC_r\left\{\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\dots\text{upto } m \text{ terms}\right\}$$

$$=\sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{1}{2^r}+\sum_{r=0}^{n}(-1)^r\cdot{}^nC_r\frac{3^r}{2^{2r}}+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{7^r}{2^{3r}}+\dots$$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\dots$$

upto m terms

$$=\frac{1}{2^n}+\frac{1}{2^{2n}}+\frac{1}{2^{3n}}\dots \text{ upto } m \text{ terms}$$

$$=\frac{\frac{1}{2^n}\left\{1-\left(\frac{1}{2^n}\right)^m\right\}}{\left(1-\frac{1}{2^n}\right)}=\frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

10 We have,

$$\left[2^{\log_2\sqrt{9^{x-1}+7}}+\frac{1}{2^{(1/5)\log_2(3^{x-1}+1)}}\right]^7$$

$$=\left[\sqrt{9^{x-1}+7}+\frac{1}{(3^{x-1}+1)^{1/5}}\right]^7$$

$$\therefore T_6={}^7C_5(\sqrt{9^{x-1}+7})^{7-5}\left[\frac{1}{(3^{x-1}+1)^{1/5}}\right]^5$$

$$={}^7C_5(9^{x-1}+7)\frac{1}{(3^{x-1}+1)}$$

$$\Rightarrow\quad 84={}^7C_5\frac{(9^{x-1}+7)}{(3^{x-1}+1)}$$

$$\Rightarrow\quad 9^{x-1}+7=4(3^{x-1}+1)$$

$$\Rightarrow\quad \frac{3^{2x}}{9}+7=4\left(\frac{3^x}{3}+1\right)$$

$$\Rightarrow\quad 3^{2x}-12(3^x)+27=0$$

$$\Rightarrow\quad y^2-12y+27=0 \quad (\text{put } y=3^x)$$

$$\Rightarrow\quad (y-3)(y-9)=0$$

$$\Rightarrow\quad y=3,9$$

$$\Rightarrow\quad 3^x=3,9$$

$$\Rightarrow\quad x=1,2$$

11 Last term of $\left(2^{1/3}-\frac{1}{\sqrt{2}}\right)^n$ is

$$T_{n+1}={}^nC_n(2^{1/3})^{n-n}\left(-\frac{1}{\sqrt{2}}\right)^n$$

$$={}^nC_n(-1)^n\frac{1}{2^{n/2}}=\frac{(-1)^n}{2^{n/2}}$$

Also, we have

$$\left(\frac{1}{3^{5/3}}\right)^{\log_3 8}=3^{-(5/3)\log_3 2^3}=2^{-5}$$

Thus, $\frac{(-1)^n}{2^{n/2}}=2^{-5}\Rightarrow\frac{(-1)^n}{2^{n/2}}=\frac{(-1)^{10}}{2^5}$

$$\Rightarrow\quad \frac{n}{2}=5\Rightarrow n=10$$

Now, $T_5=T_{4+1}={}^{10}C_4(2^{1/3})^{10-4}\left(-\frac{1}{\sqrt{2}}\right)^4$

$$=\frac{10!}{4!6!}(2^{1/3})^6(-1)^4(2^{-1/2})^4$$

$$=210(2)^2(1)(2^{-2})=210$$

12 **Key idea** $=(a+b)^n+(a-b)^n$

$$=2({}^nC_0a^n+{}^nC_2a^{n-2}b^2+{}^nC_4a^{n-4}b^4\dots)$$

We have

$$(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5, x>1$$

$$=2({}^5C_0x^5+{}^5C_2x^3(\sqrt{x^3-1})^2+{}^5C_4x(\sqrt{x^3-1})^4)$$

$$=2(x^5+10x^3(x^3-1)+5x(x^3-1)^2)$$

$$=2(x^5+10x^6-10x^3+5x^7-10x^4+5x)$$

Sum of coefficients of all odd degree terms is

$$2(1-10+5+5)=2$$

13 The general term in the expansion of $\left(x\cos\alpha+\frac{\sin\alpha}{x}\right)^{20}$ is

$${}^{20}C_r(x\cos\alpha)^{20-r}\left(\frac{\sin\alpha}{x}\right)^r$$

$$={}^{20}C_rx^{20-2r}(\cos\alpha)^{20-r}(\sin\alpha)^r$$

For this term to be independent of x, we get

$$20-2r=0$$

$$\Rightarrow\quad r=10$$

Let β = Term indeoendent of x

$$={}^{20}C_{10}(\cos\alpha)^{10}(\sin\alpha)^{10}$$

$$={}^{20}C_{10}(\cos\alpha\sin\alpha)^{10}$$

$$={}^{20}C_{10}\left(\frac{\sin 2\alpha}{2}\right)^{10}$$

Thus, the greatest possible value of β is ${}^{20}C_{10}\left(\frac{1}{2}\right)^{10}$.

14 Let $P(n)=(n)^7-n$

By mathematical induction

For $n=1$,

$P(1)=0$, which is divisible by 7.

For $n=k$

$$P(k)=k^7-k$$

Let $P(k)$ be divisible by 7.

$\therefore\quad k^7-k=7\lambda$, for some $\lambda\in N$ ... (i)

For $n=k+1$,

$$P(k+1)=(k+1)^7-(k+1)$$

$$=({}^7C_0k^7+{}^7C_1k^6+{}^7C_2k^5+\dots+{}^7C_6\cdot k+{}^7C_7)-(k+1)$$

$$=(k^7-k)+7\{k^6+3k^5+\dots+k\}$$

$$=7\lambda+7\{k^6+3k^5+\dots+k\} \text{[Using Eq. (i)]}$$

$\Rightarrow$ Divisible by 7.

So, both statements are true and Statement II is correct explanation of Statement I.

15 We know that, in the expansion of $(a+b)^n$, pth term from the end is $(n-p+2)$th term from the beginning.

So, 5th term from the end is

$=(n-5+2)$ th term from the beginning

$=(n-3)$ th term from the beginning

$=(n-4+1)$ th term from the beginning ...(i)

$\therefore$ We have,

$$\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n=\left(2^{1/4}+\frac{1}{3^{1/4}}\right)^n$$

Now, 5th term from the beginning is

$$T_{4+1}={}^nC_4\,(2^{1/4})^{n-4}\,(3^{-1/4})^4$$

$$={}^nC_4\,2^{\frac{n-4}{4}}\,3^{-1} \quad \dots\text{(ii)}$$

And 5th term from the end is

$$T_{(n-4)+1}={}^nC_{n-4}(2^{1/4})^{n-n+4}(3^{1/4})^{n-4}$$

$$={}^nC_4\,2\cdot 3^{-\frac{n+4}{4}}$$

$$[\because {}^nC_r={}^nC_{n-r}] \quad \dots\text{(iii)}$$

So, from the given condition, we have

$$\frac{\text{Fifth term from the beginning}}{\text{Fifth term from the end}}=\frac{\sqrt{6}}{1}$$

$$\Rightarrow\quad \frac{{}^nC_4\cdot 2^{\frac{n-4}{4}}\cdot 3^{-1}}{{}^nC_4\cdot 2\cdot 3^{\frac{-n+4}{4}}}=\frac{\sqrt{6}}{1}$$

$$\Rightarrow\quad 2^{\frac{n-4}{4}-1}\cdot 3^{-1-\left(\frac{4-n}{4}\right)}=\sqrt{6}$$

$$\Rightarrow\quad 2^{\frac{n-8}{4}}\,3^{\frac{n-8}{4}}=6^{1/2}$$

$$\Rightarrow\quad (2\times 3)^{\frac{n-8}{4}}=(2\cdot 3)^{1/2}$$

$$\Rightarrow\quad \frac{n-8}{4}=1/2$$

$$\Rightarrow\quad n=2+8 \quad \therefore n=10$$